An approximate solution of the same problem in plane extension was given by Bowie.[6] As a special case of interest, when $b = 0$, Eq. (11) reduces to $\tau^{\infty} a^{1/2} \sin\alpha$, which is the solution of a single crack of length $2a$ under longitudinal shear. This limiting case also may be obtained upon substituting Eqs. (4) and (10) into Eq. (8) with $R = a/2$.

In closing, it should be pointed out that, in contrast to the complex variable method used in the plane theory of elasticity, the problem in longitudinal shear permits irrationality of the mapping function. Consequently, the present approach will allow the consideration of crack configurations heretofore avoided.

References

[1] Sih, G. C., "On the crack-tip stress-intensity factors for cylindrical bars under torsion," J. Aerospace Sci. **29**, 1265–1266 (1962).

[2] Sih, G. C., "Flexural stress distribution near a sharp crack," AIAA J. **1**, 492–493 (1963).

[3] Sih, G. C., "Strength of stress singularities at crack tips for flexural and torsional problems," J. Appl. Mech., Trans. Am. Soc. Mech. Engrs. Paper 63-APMW-8 (1963).

[4] Irwin, G. R., "Fracture mechanics," *Structural Mechanics* (Pergamon Press, New York, 1960), pp. 557–592.

[5] Milne-Thomson, L. M., *Theoretical Hydrodynamics* (Macmillan Co. Ltd., London, 1949), 2nd ed., Chap. VI, p. 156.

[6] Bowie, O. L., "Analysis of an infinite plate containing radial cracks originating at the boundary of an internal circular hole," J. Math. Phys. **25**, 60–71 (1956).

Effect of Nodal Regression on Spin-Stabilized Communication Satellites

B. Paul* and E. Y. Yu*

Bell Telephone Laboratories Inc., Murray Hill, N. J.

A SPIN-STABILIZED satellite is useful for communication purposes if the antenna pattern is symmetric about the spin axis as in a so-called toroidal antenna where the unilluminated region is a cone about the spin axis with semivertex angle β, as shown in Fig. 1. The illuminated space consists of a triangle of revolution with an "antenna angle" 2α defined as shown in Fig. 1 by $\alpha = 90° - \beta$. The angle α will be referred to as the "semivertex angle" of the antenna. If the spin axis were to remain perpendicular† to the orbital plane at all times, the angle α needs never exceed the minimum value α_1, where (see Fig. 1)

$$\sin\alpha_1 = R_0/r \tag{1}$$

where R_0 is the earth radius and r is the geocentric satellite distance.

However, if the spin axis is initially perpendicular to the orbital plane, it will not remain so, since the orbit regresses while the spin-axis direction remains fixed in inertial space.‡ Thus, the actual antenna angle 2α must be increased by a certain tolerance angle $2\alpha_T$ in order for the antenna pattern to maintain full coverage of the visible earth surface. The authors now show how to find α_T for a given circular orbit.

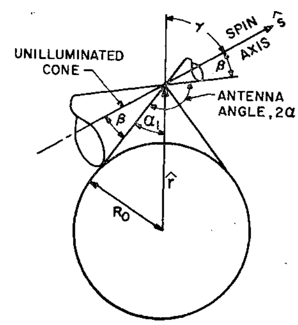

Fig. 1 Antenna shown casting no shadows.

Figure 2 shows, on a unit sphere, the original orbit of a satellite at inclination i with the equator. The pole of the original orbit is at S, and the unit vector $\hat{s} = \mathbf{OS}$ is parallel to the spin axis of the satellite which is chosen to be perpendicular to the original orbital plane. As time goes on, the node (intersection of orbital plane with earth equator) shifts from its original position N_1 through the angle Ω to a new position N_2. At the same time, the satellite may be anywhere in the perturbed orbit such as at position Q that has radius vector $\hat{r}$ from the center of the earth. The radius vector $\hat{r}$ lying in the orbital plane is normal to $\hat{p}$, the perpendicular to the perturbed orbit. The tip P of vector $\hat{p}$ moves through an angle Ω along the small circle at colatitude i.

In order to find the antenna half-angle α required for a given orbit, imagine a plane passed through the radius vector r and the spin vector $\hat{s}$, as shown in Fig. 1, from which it is evident that

$$\beta = \gamma - \alpha_1 \tag{2}$$

and

$$\alpha = 90° - \beta = 90° + \alpha_1 - \gamma \tag{3}$$

where γ is the acute angle between $\hat{r}$ and $\hat{s}$. The maximum value of α required anywhere in the orbit occurs where γ is minimum. From Fig. 2, it may be seen that the minimum value of γ occurs when $\hat{r}$ is in the plane of $\hat{p}$ and $\hat{s}$, at which point $\gamma = \gamma_{\min}$ and

$$\gamma_{\min} = 90° - \delta \tag{4}$$

or

$$\sin\gamma_{\min} = \cos\delta = \hat{p}\cdot\hat{s} \tag{5}$$

In order to calculate δ, which is a sector of the great circle, through P and S, one may apply the cosine law of spherical trigonometry to the spherical triangle APS (A is the earth's pole) and find

$$\cos\delta = \cos^2 i + \sin^2 i \cos\Omega \tag{6}$$

The final expression for the required antenna half-angle is given by

$$\alpha = 90° + \alpha_1 - \text{arc}\,\sin(\cos^2 i + \sin^2 i \cos\Omega) \tag{7}$$

This expression is valid so long as α remains less than 90°. For values of i and Ω such that Eq. (7) predicts α greater than 90°, one must use $\alpha = 90°$ corresponding to an isotropic antenna. This may be verified from Fig. 1, which shows that when γ is less than α_1 the spin axis intercepts the earth, and the entire earth cannot be illuminated except by an isotropic antenna with $\alpha = 90°$. Thus, the condition for which a toroidal antenna can provide more intense illumination than an isotropic antenna with equal total power and complete earth coverage at all times is that δ is less than $90° - \alpha_1$, or

$$\cos\delta = \cos^2 i + \sin^2 i \cos\Omega > \sin\alpha_1 \tag{8}$$

Received August 2, 1963.

* Formerly Member of Technical Staff; now Head, Solid Mechanics Research, Ingersoll-Rand Research and Development Laboratory, Bedminster, N. J. Member AIAA.

† This is not the only useful type of spin-axis orientation. For example, the Telstar satellite is not oriented in this manner. However, because of its potentiality for increase in gain, this method of orientation merits some study.

‡ Precession of the spin axis due to the interaction between the geomagnetic field and the magnetic dipole moment of the satellite will require an additional tolerance on α but is not considered here.

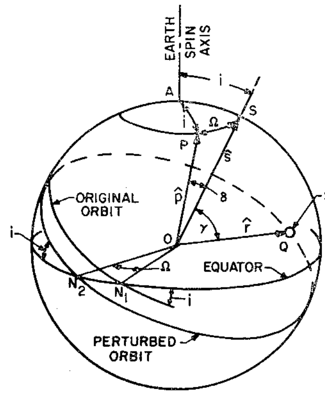

Fig. 2 Unit sphere showing original and perturbed orbits.

For a fixed value of α_1, it is seen that this inequality may be satisfied only for values of i below a certain critical value. To find this critical value, note that the left-hand side of inequality (8) is a minimum when $\Omega = 180°$, so that the critical value of i occurs at

$$\cos^2 i - \sin^2 i = \cos 2i = \sin\alpha_1 \quad (9)$$

This last result shows that an isotropic antenna is necessary if $\cos 2i < \sin\alpha_1 = \cos(90° - \alpha_1)$, or if

$$i > 45° - (\alpha_1/2) \quad (10)$$

This shows that isotropic antennas are required for all inclinations over 45° and many orbits below 45°, depending on altitude. For example, an altitude of 6000 miles above the earth's surface corresponds to $\alpha_1 = 23.5°$. Thus, all orbits at this altitude with an inclination in excess of $i = 45° - 11.75° = 33.25°$ require an isotopic antenna for complete earth coverage at all times. For lower inclinations, the required angle may be found from Eq. (7) or from Fig. 3, which shows how the required tolerance angle $\alpha_T = \alpha - \alpha_1$ varies with inclination and regression of the node Ω.

Finally, from Eq. (7), it is seen that (if $i < 45° - \alpha_1/2$) the maximum value of α occurs when Ω is 180°, and the required antenna angle is given by

$$2\alpha_{\max} = 2[90° + \alpha_1 - \text{arc}\sin(\cos 2i)] = 2\alpha_1 + 4i \quad (11)$$

and $2\alpha_{\max} = 180°$ for $i > 45° - \alpha_1/2$.

For nearly polar orbits, the regression rate of the orbital plane is quite low, since it is proportional to the cosine of the inclination of the orbital plane.[1] For 6000-mile orbits the regression rate is only 2.5°/yr/deg off the polar orbital plane. Therefore, for satellites of finite service life of, for example, 5–10 yr, it is possible to acquire both antenna gain and complete earth coverage. To show this in a specific way on 6000-mile nearly polar orbits, Eq. (7) is plotted in Fig. 4, taking into account an initial injection tolerance (assumed to be 2°). It is noted from Fig. 4 that for orbits within 5° from the polar plane the toroidal antenna pattern will continue to illuminate the visible earth for at least 5 yrs.

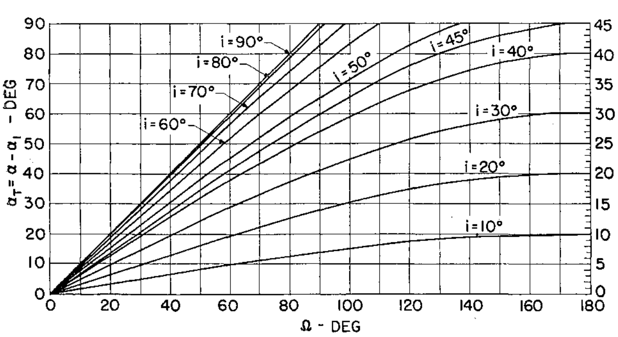

Fig. 3 Tolerance angle α_T vs position of orbital node Ω for various angles of inclination i.

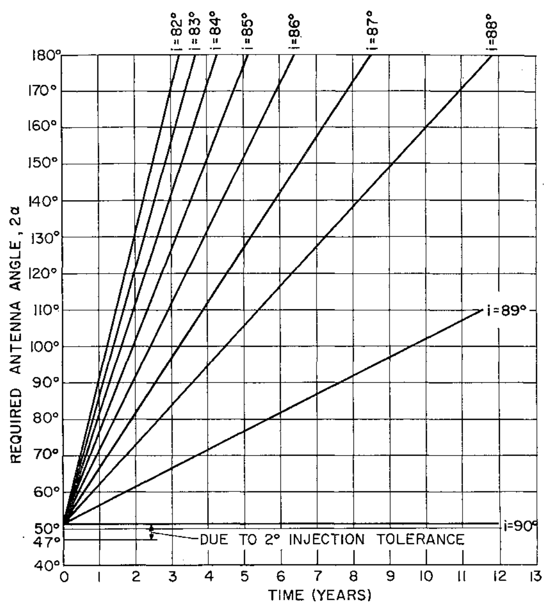

Fig. 4 Required antenna angle for complete coverage at 6000-mile altitude and various orbital inclinations, including effect of 2° injection error and orbital precession.

Reference

[1] Blanco, V. M. and McCuskey, S. W., *Basic Physics of the Solar System* (Addison-Wesley Publishing Co. Inc., Reading, Mass., 1961), p. 202.

Mach Number Independence of the Conical Shock Pressure Coefficient

Glen W. Zumwalt* and H. H. Tang†
Oklahoma State University, Stillwater, Okla.

Nomenclature

C_p = pressure coefficient
K = constant
M = Mach number
P = static pressure
q = dynamic pressure
ϵ = density ratio [Eq. (2)]
θ = conical angle (semivertex)
γ = specific heat ratio

Subscripts

∞ = freestream conditions
c = conditions on the cone surface
w = conditions immediately downstream of the conical shock wave

A SIMPLE relationship exists for conical shocks which, to the authors' knowledge, has not been pointed out before and for which no complete explanation is readily found. This relationship concerns the pressure coefficient across a conical shock produced by air (a perfect gas) flowing at zero

Received March 27, 1963; revision received July 15, 1963. The work reported in this paper was performed in connection with research sponsored by Sandia Corporation, Albuquerque, N. Mex.
* Associate Professor of Mechanical and Aerospace Engineering. Member AIAA.
† Instructor in Aerospace Engineering. Member AIAA.